11-15-21

Last Time: Curl and Divergence

$\text{curl}(\vec{v}) = \nabla \times \vec{v}$ | $\text{div}(\vec{v}) = \nabla \cdot \vec{v}$

$\uparrow$ $\langle P, Q, R \rangle$

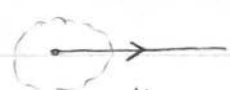
Prop: ① $\text{curl}(\nabla f) = \vec{0}$ ② $\text{div}(\text{curl}(\vec{v})) = 0$

Interpretations of Curl and Divergence

① Curl measures "how swirly is the v.f."?

$\hookrightarrow$ $\text{curl}(\vec{v})$ is always "swirly".

② divergence measures "does the v.f. tend to push points away from a little open region?"



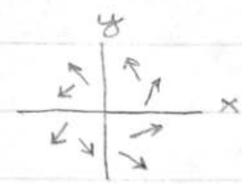
divergence $\neq 0$

$\leftarrow$ swirly...

divergence $= 0$

Ex.

Consider v.f. $\langle P(x,y), Q(x,y), 0 \rangle = \vec{v}$.



* View from above

$$\text{curl}(\vec{v}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{bmatrix}$$

$$= \langle -Q_z, +P_z, Q_x - P_y \rangle$$

$$= \langle 0, 0, Q_x - P_y \rangle$$

$$= \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$$

Recasting Green's Theorem w/ Vector fields:

Let $\vec{v} = \langle P(x,y), Q(x,y), 0 \rangle$, have cts partial derivatives on some open region $R \subseteq \mathbb{R}^2$ and containing closed region D w/ piecewise-smooth simple, closed boundary curve. Then ① $\iint_D \text{curl}(\vec{v}) \cdot \vec{k}\, dA = \int_{\partial D} \vec{v} \cdot d\vec{r}$ and ② $\int_{\partial D} \vec{v} \cdot (y'(t)\vec{i} - x'(t)\vec{j}) \frac{1}{|\vec{r}'(t)|}\, ds$

$= \iint_D \text{div}(\vec{v})\, dA$

Why: ① $\text{curl}(\vec{v}) = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$, so $\text{curl}(\vec{v}) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$\therefore \iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

Green's Theorem $\longrightarrow$ $= \int_{\partial D} P\,dx + Q\,dy = \int_{t=a}^{b} (P(x,y)x'(t) + Q(x,y)y'(t))\,dt$

$= \int_{t=a}^{b} \langle P, Q, 0 \rangle \cdot \langle x', y', z' \rangle \, dt$

$= \int_{\partial D} \vec{v} \cdot d\vec{r}$

② $\iint_D \text{div}(\vec{v})\,dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$ $\qquad \omega = \langle \overset{A}{-Q}, \overset{B}{P}, 0 \rangle$

$= \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y} \right) dA \qquad \leftrightarrow \iint_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dA$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$= \int_{\partial D} -Q\,dx + P\,dy \qquad \leftrightarrow \int_{\partial D} A\,dx + B\,dy$

$= \int_{t=a}^{b} (-Qx' + Py')\,dt$

$= \int_{t=a}^{b} (Py' - Qx')\,dt$

$= \int_{t=a}^{b} \langle P, Q \rangle \cdot \langle y', -x' \rangle \, dt$

$= \int_{\partial D} \vec{v} \cdot (y'(t)\vec{i} - x'(t)\vec{j}) \frac{1}{|r'(t)|} \, ds$

NB: These two ways of rewriting Green's Theorem w/

① Curl & ② divergence

are jumping points for generalizing Green's Theorem

Stokes's Theorem $\qquad\qquad$ Divergence Theorem

Not on Exam 3, but on the final

16.6: Parametric Surfaces

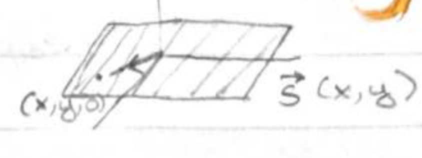
Idea: Generalize space curves to have dimension 2...

Def: A parametric surface in 3-space is given by vector function

$\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ on some domain $D \subseteq \mathbb{R}^2$.

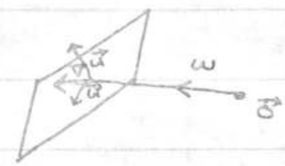
Ex. The Euclidean Plane sits on $\mathbb{R}^3$ as a parametric surface

$\vec{S}(x,y) = \langle x, y, 0 \rangle$



Ex.

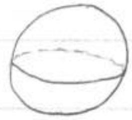
Every plane Π in $\mathbb{R}^3$ can be parameterized by
$\vec{S}(a,b) = a\vec{u} + b\vec{v} + \vec{\omega}$ for suitable vectors $\vec{u}, \vec{v}, \vec{\omega}$. on $D = \mathbb{R}^2$



I.e. $\vec{S}(a,b) = \langle u_1 a + v_1 b + \omega_1, u_2 a + v_2 b + \omega_2, u_3 a + v_3 b + \omega_3 \rangle$.

Ex.

The sphere of radius $r > 0$ is parameterized by
$\vec{S}(\theta, \varphi) = \langle r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi) \rangle$ on $D = [0, 2\pi] \times [0, \pi]$.



Ex.

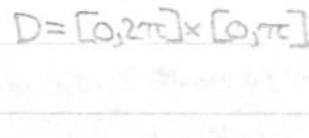
The torus has parameterization
$\vec{S}(\theta, \varphi) = \langle (2+\sin(\theta))\cos(\varphi), (2+\sin(\theta))\sin(\varphi), \cos(\theta) \rangle$
on $D = [0, 4\pi] \times [0, 2\pi]$

Donut, aka torus

Ex.

Parametrize the paraboloid $z = x^2 + 2y^2$.

NB: There is no one parameterization of a surface ...



Sol. ①: $\vec{S}(x,y) = \langle x, y, x^2 + 2y^2 \rangle$ on $D = \mathbb{R}^2$

Sol. ②: $\vec{S}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), (r\cos(\theta))^2 + 2(r\sin(\theta))^2 \rangle$

$= \langle r\cos(\theta), r\sin(\theta), r^2(1 + \sin^2(\theta)) \rangle$ on $D = [0, \infty) \times [0, 2\pi]$

Sol. ③: $\vec{S}(r, \theta) = \langle \sqrt{2}\, r\cos(\theta), r\sin(\theta), 2r^2 \rangle$ on $D = [0, \infty) \times [0, 2\pi]$

Ex.

A surface of revolution (about x-axis) be obtained for a function $f(x)$ via

$\vec{s}(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta) \rangle$

on $D = \mathrm{dom}(f) \times [0, 2\pi]$.